

# MAKING INFERENCE FOR RELIABILITY INDICES BASED ON BETA BINOMIAL MODEL WITH SAS SOFTWARE

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## Abstract

When measuring characteristics such as mastery of course, clinical diagnosis or patient satisfaction, the reliability needs to be assessed. The raw agreement index and kappa index are two important indices among the reliability indices.

The estimates of these two indices could be obtained based on the distribution which the data follow. This paper discusses the estimation of these reliability indices and their variances based on the beta-binomial distribution so that the inference for the indices could be made.

To complete the computation, this paper also discusses an efficient algorithm to obtain the partial derivatives of the univariate and bivariate beta-binomial density functions with respect to their parameters. The results in the computation could be obtained using SAS functions and codes.

## 1. Introduction

When we wish to measure characteristics such as mastery of course, clinical diagnosis or patient satisfaction, we usually develop a scale or test that is composed of variety of related items. The responses to each of the items could be graded and summed, resulting in a score for each case or respondent. Then a criterion-standard may be used to classify the respondents in two or more categories.

The reliability would be most appropriately viewed as classification consistency across repeated test administrations using the same form or two equivalent forms. The consistency may be quantified by the raw agreement index  $p$  which expresses the proportion of respondents classified in the same category by both testing. Moreover, another index,  $k$  coefficient, could express the extent of improvement in classification consistency which is reflected by the dependency between two equivalent sets of data.

The estimates of these two indices could be obtained based on the distribution which the test scores follow. Huynh (1978) proposed that the test scores on one form followed a beta-binomial model and test scores on both forms distribute jointly as a bivariate beta-

binomial distribution.

## 2. Beta-binomial density functions and their derivatives

Let  $X$  be the test score obtained by administering an  $n$ -item test to each respondent in the population and follow the univariate beta-binomial distribution with parameters  $\alpha$  and  $\beta$ . The density function of the beta-binomial distribution is

$$f(x) = \binom{n}{x} B(\alpha+x, n+\beta-x) / B(\alpha, \beta). \quad (2.1)$$

Let  $X$  and  $Y$  be the test scores obtained by administering two equivalent  $n$ -item tests to each respondent in the population and follow the bivariate beta-binomial distribution with parameters  $\alpha$  and  $\beta$ . The density function of the bivariate beta-binomial distribution is

$$f(x, y) = \binom{n}{x} \binom{n}{y} \frac{B(\alpha+x+y, 2n+\beta-x-y)}{B(\alpha, \beta)}. \quad (2.2)$$

Using the relation between the Beta function and Gamma function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (2.3)$$

the density functions in (2.1) and (2.2) could be rewritten as

$$f(x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+x)\Gamma(n+\beta-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)} \quad (2.4)$$

and

$$f(x, y) = \binom{n}{x} \binom{n}{y} \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+x+y)}{\Gamma(2n+\beta-x-y)\Gamma(\alpha)\Gamma(\beta)\Gamma(2n+\alpha+\beta)}. \quad (2.5)$$

To obtain the partial derivatives of the

density functions with respect to the parameters  $\alpha$  and  $\beta$ , we may use the following inductive formulas:

$$f(0) = \prod_{i=1}^n \frac{n+\beta-i}{n+\alpha+\beta-i} \quad (2.6)$$

and

$$f(x+1) = f(x) \frac{(n-x)(\alpha+x)}{(x+1)(n+\beta-x-1)}, \quad x=0, 1, \dots, n-1; \quad (2.7)$$

$$f(0,0) = \prod_{i=1}^{2n} \frac{2n+\beta-i}{2n+\alpha+\beta-i} = f(0)R \quad (2.8)$$

$$\text{where } R = \prod_{i=1}^n \frac{2n+\beta-i}{2n+\alpha+\beta-i}$$

and

$$f(x+1, y) = f(x, y) \frac{(n-x)(\alpha+x+y)}{(x+1)(2n+\beta-x-y-1)}, \quad x=0, 1, \dots, n-1, y=0, 1, \dots, n; \quad (2.9)$$

Therefore, we have the following recurrence approach to obtain the derivatives with respect to  $\alpha$ :

$$\frac{\partial f(0)}{\partial \alpha} = f(0) \sum_{i=1}^n \frac{-1}{n+\alpha+\beta-i} \quad (2.10)$$

$$\frac{\partial f(x+1)}{\partial \alpha} = \frac{n-x}{(x+1)(n+\beta-x-1)} [(\alpha+x) \frac{\partial f(x)}{\partial \alpha} + f(x)]; \quad (2.11)$$

$$\frac{\partial f(0,0)}{\partial \alpha} = R \left[ \frac{\partial f(0)}{\partial \alpha} + f(0) \sum_{i=1}^n \frac{-1}{2n+\alpha+\beta-i} \right] \quad (2.12)$$

and

$$\frac{\partial f(x+1, y)}{\partial \alpha} = C(x) R_{\alpha}(x, y) \quad (2.13)$$

$$\frac{\partial f(x, y+1)}{\partial \alpha} = C(y) R_{\alpha}(x, y) \quad (2.14)$$

where

$$C(t) = \frac{n-t}{t+1}, \quad R_{\alpha}(x, y) = \frac{1}{2n+\beta-x-y-1} \left[ \frac{\partial f(x, y)}{\partial \alpha} (\alpha+x+y) + f(x, y) \right]. \quad (2.15)$$

We also have the similar recurrence approach to the partial derivatives with respect to  $\beta$ :

$$\frac{\partial f(0)}{\partial \beta} = f(0) \sum_{i=1}^n \frac{\alpha-2i}{(n+\alpha+\beta-i)(n+\beta+i)} \quad (2.16)$$

and

$$\frac{\partial f(x+1)}{\partial \beta} = \frac{(n-x)(\alpha+x)}{(x+1)(n+\beta-x-1)} \left[ \frac{\partial f(x)}{\partial \beta} - \frac{1}{n+\beta-x-1} f(x) \right]; \quad (2.17)$$

$$\frac{\partial f(0,0)}{\partial \beta} = R \left[ \frac{\partial f(0)}{\partial \beta} + f(0) \sum_{i=1}^n \frac{\alpha}{(2n+\alpha+\beta-i)(2n+\beta-i)} \right] \quad (2.18)$$

and

$$\frac{\partial f(x+1, y)}{\partial \beta} = C(x) R_{\beta}(x, y) \quad (2.19)$$

$$\frac{\partial f(x, y+1)}{\partial \beta} = C(y) R_{\beta}(x, y) \quad (2.20)$$

where

$$C(t) = \frac{n-t}{t+1},$$

$$R_{\beta}(x, y) = \frac{(\alpha+x+y)}{2n+\beta-x-y-1}$$

$$\left[ \frac{\partial f(x, y)}{\partial \beta} - \frac{f(x, y)}{2n+\beta-x-y-1} \right]. \quad (2.21)$$

### 3. Estimation of p and k and their standard errors

Assuming that the responses would be classified into l categories defined by l-1 integers  $C_i$ ,  $i=1, 2, \dots, l-1$  with  $0 < C_1 < \dots < C_{l-1} < n$ , the raw agreement index could be obtained as

$$p = \sum_{i=1}^l \left[ \sum_{x, y=C_{i-1}}^{C_i-1} f(x, y) \right] \quad (3.1)$$

Here  $C_0 = 0$  and  $C_{l-1} = n+1$ . The lower limit for classification consistency is given as

$$p_c = \sum_{i=1}^l \left[ \sum_{x=C_{i-1}}^{C_i-1} f(x) \right]^2 \quad (3.2)$$

Then the kappa index would be obtained as

$$\kappa = (p - p_c) / (1 - p_c) \quad (3.3)$$

To estimate p and k, we first need to obtain the moment estimates  $\alpha$  and  $\beta$  as follows:

$$\hat{\alpha} = (-1 + 1/\hat{\alpha}_{21}) \bar{X} \quad (3.4)$$

and

$$\hat{\beta} = -\hat{\alpha} + n/\hat{\alpha}_{21} - n \quad (3.5)$$

where

$$\hat{\alpha}_{21} = \frac{n}{n-1} \left( 1 - \frac{\bar{X}(n-\bar{X})}{nS^2} \right), \quad (3.6)$$

here  $\bar{x}$  and  $s$  are the mean and standard error of the test scores of the respondents.

As for the asymptotic sampling distribution of these estimates, we have the following result that  $\sqrt{m} (\hat{\alpha} - \alpha, \hat{\beta} - \beta)$  asymptotically follows a bivariate normal distribution with zero mean and covariance matrix  $\Sigma = A^{-1}$ , when  $m$ , sample size, is large enough, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$a_{11} = \sum_{x=0}^n \left[ \frac{\partial f(x)}{\partial \alpha} \right]^2 / f(x)$$

$$a_{12} = a_{21} = \sum_{x=0}^n \frac{\partial f(x)}{\partial \alpha} \frac{\partial f(x)}{\partial \beta} / f(x)$$

$$a_{22} = \sum_{x=0}^n \left[ \frac{\partial f(x)}{\partial \beta} \right]^2 / f(x). \quad (3.7)$$

Moreover we have the following result that when  $m$ , sample size, is large enough,  $\sqrt{m} (\hat{p} - p, \hat{\kappa} - \kappa)$  asymptotically follow a bivariate normal distribution with zero mean and covariance matrix

$$\Omega = \begin{pmatrix} \text{Var}(p) & \text{Cov}(p, \kappa) \\ \text{Cov}(p, \kappa) & \text{Var}(\kappa) \end{pmatrix}. \quad (3.8)$$

and

$$\Omega = \Phi' \Sigma \Phi. \quad (3.9)$$

where

$$\Phi' = \begin{pmatrix} \frac{\partial p}{\partial \alpha} & \frac{\partial p}{\partial \beta} \\ \frac{\partial \kappa}{\partial \alpha} & \frac{\partial \kappa}{\partial \beta} \end{pmatrix}. \quad (3.10)$$

### 4. Discussion

Now we could make inference on p and k using their estimates and estimated standard errors. All the approach could be

accomplished by SAS functions and codes.

As illustrated above, the estimates of  $p$  and  $k$  could be obtained directly from SAS GAMMA function. Then, the recurrence approach to compute the derivatives of the beta-binomial density function with respect to their parameters could be completed by repeatedly using SAS programs.

#### References

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